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CONTROL OF A SYSTEM WITH ONE DEGREE OF FREEDOM UNDER COMPLEX RESTRICTIONS†

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A very simple dynamical system with one degree of freedom, controlled by a force of bounded magnitude, is considered. It is assumed that the magnitude of the force may increase gradually at a finite rate and that the force is switched off instantaneously. Under these restrictions, which simulate real servo-systems, a control is constructed that steers the system to the origin and has the simplest possible structure. © 2000 Elsevier Science Ltd. All rights reserved.

1. FORMULATION OF THE PROBLEM

The simplest dynamical system with one degree of freedom, described by Newton's Second Law, has often served as a model in control theory [1, 2]. On the one hand, this model has been used to work out and demonstrate schemes and methods for solving control problems. On the other, a system with one degree of freedom has been used as an element in certain schemes for decomposing non-linear systems with many degrees of freedom into simpler subsystems [3, 4].

Consider a system with one degree of freedom of the form

$$m\ddot{\xi} = F \tag{1.1}$$

where ξ is a coordinate, m is the constant mass of the system, the dots denote differentiation with respect to time t and F is a controlling force subject to the restriction

$$|F| \leqslant F_0 \tag{1.2}$$

where F_0 is a constant. Let us consider the problem of bringing system (1.1) to the origin of the phase plane, that is, to the state $\xi = \dot{\xi} = 0$. If no restrictions other than (1.2) are imposed on the system, then a time-optimal control solving this problem is well known [1] and is a bang–bang control with at most one switching. The typical time-dependence of this control is shown in Fig. 1. The control forces may be switched on and off instantaneously.

In actual servo-systems that implement control impulses, instantaneous variation of a force is frequently impossible. If allowance is made for the fact that the magnitude of the force may change at a finite rate, we arrive at the restriction

$$|\dot{F}| \leqslant \upsilon_0 \tag{1.3}$$

where υ_0 is a constant.

A solution of the problem of synthesizing a time-optimal control for system (1.1) subject to condition (1.3) for the zero terminal state $\xi = \dot{\xi} = 0$ was obtained in [5]. In that solution, restriction (1.2) on the magnitude of the force was not taken into consideration, that is, it was assumed that it was not achieved. As to the time-optimal control problem for system (1.1) taking both restrictions (1.2) and (1.3) into account, we have not come upon any solution in the literature.

The control problem for system (1.1) will be treated in the following formulation. It is assumed that the control force is bounded as in (1.2), while condition (1.3) will only be satisfied when the magnitude of the force increases, that is, when $d|F|/dt > 0$. At the same time, the force may be switched off instantaneously. These restrictions may be written as a system of inequalities

$$|F| \leqslant F_0; \quad \dot{F} \leqslant \upsilon_0 \ \text{ if } \ F \geqslant 0, \qquad \dot{F} \geqslant -\upsilon_0 \ \text{ if } \ F \leqslant 0 \tag{1.4}$$

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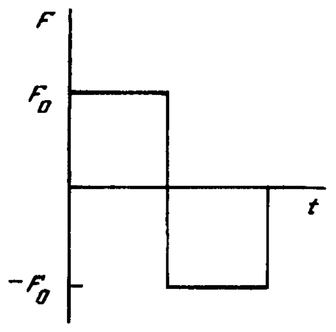

Fig. 1.

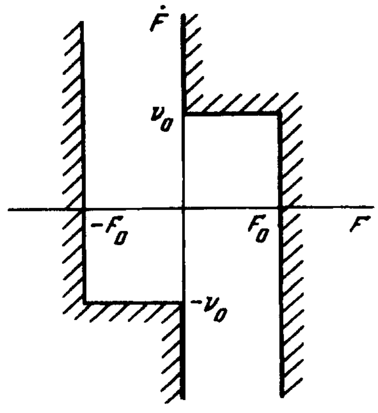

Fig. 2.

For the domain defined by inequalities (1.4) in the $(F, \dot{F})$ plane, see Fig. 2.

Conditions (1.4) simulate the following situation: the control force may be increased only gradually, at a finite rate, but it can be switched off instantaneously. This is not infrequently the case in practice, since deceleration is often implemented by means other than acceleration.

We introduce the following dimensionless variables

$$t' = v_0 F_0^{-1} t, \quad x = m v_0^2 F_0^{-3} \xi, \quad y = m v_0 F_0^{-2} \dot{\xi}, \quad z = F_0^{-1} F, \quad u = v_0^{-1} \dot{F} \tag{1.5}$$

Equation (1.1) and conditions (1.4) take the following form in terms of these new variables

$$\dot{x} = y, \quad \dot{y} = z, \quad \dot{z} = u \tag{1.6}$$

$$|z| \leqslant 1; \quad u \leqslant 1 \text{ if } z \geqslant 0, \quad u \geqslant -1 \text{ if } z \leqslant 0 \tag{1.7}$$

Throughout this paper, dots denote differentiation with respect to the new (dimensionless) time. The prime indicating dimensionless time will be omitted from now on.

The initial conditions for system (1.6) are

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = 0 \tag{1.8}$$

and the final state is

$$x(T) = 0, \quad y(T) = 0 \tag{1.9}$$

Note that the value of $z(T)$ at a finite time may always be made equal to zero by adjusting the jump of the force $z(t)$ at time $t = T$, which is done by conditions (1.7). We may therefore assume throughout that $z(T) = 0$.

We now formulate the following problem.

It is required to find a control $u(t)$ and the corresponding trajectory, that is, functions $x(t)$, $y(t)$ and $z(t)$ satisfying Eqs (1.6), conditions (1.7), initial conditions (1.8) and final conditions (1.9) at some (non-fixed) time $T > 0$.

Henceforth we will construct a solution that solves the problem and has the simplest structure satisfying conditions (1.7). This control is obviously time-optimal.

2. TYPES OF SOLUTION

The possible forms of variation of the dimensionless force $z(t)$ are shown in Fig. 3. The figure shows the intervals in which the force increases or decreases gradually, $\dot{z} = \pm 1$ and the intervals over which the force is constant, $z = \pm 1$. These forms have the following properties.

1. At the beginning of the process, $z(0) = 0$, in accordance with initial conditions (1.8).

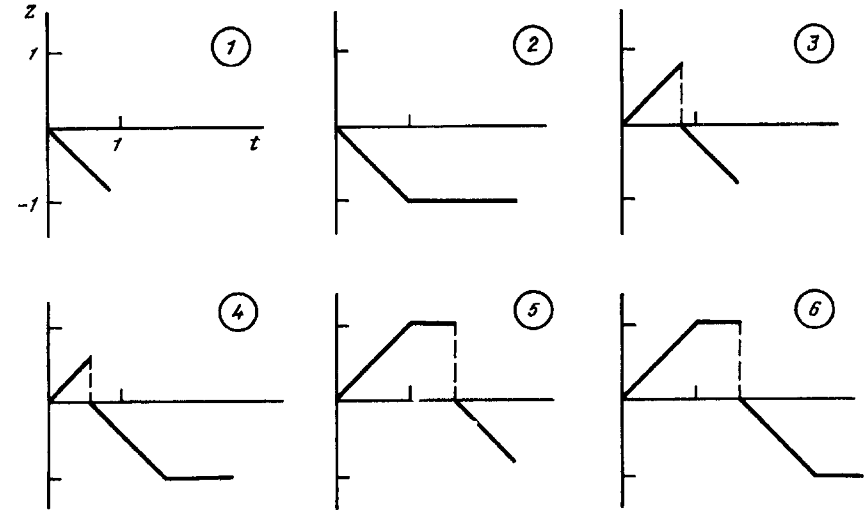

Fig. 3.

2. Forms 1–6 satisfy conditions (1.7).

3. Just before the end of the process, $z(t) < 0$ as $t \to T$. This condition is assumed for definiteness and does not affect the generality of our solution, since, besides the forms illustrated in Fig. 3, we might have similarly considered their mirror images in the t axis, namely, $z'(t) = -z(t)$.

4. The variations shown in Fig. 3 have at most one jump and one change in the sign of the force $z(t)$.

5. Form 6 of Fig. 3 is a direct extension of the bang-bang form of Fig. 1 to the case of gradual increase of the magnitude of the force, that is, to the case of conditions (1.7).

6. Forms 1–5 of Fig. 3 are special cases of 6. Indeed, in 5 the bound $z = -1$ is not achieved; in 4 the bound $z = 1$ is not achieved; in 3 neither bound $z = \pm 1$ is achieved; in 1 and 2 there is no jump in the function $z(t)$, the bound $z = -1$ being achieved in 2 but not in 1.

As will be shown below, using forms of variation of types 1–6 for $z(t)$, as well as their mirror-image laws $z'(t) = -z(t)$, one can steer system (1.6) from any initial state (1.8) to the final state (1.9).

We now introduce a domain D in the xy plane, define by the inequalities

$$D = \left\{(x, y): \begin{array}{ll} x < -\varphi(-y) & \text{if } y \leqslant 0 \\ x \leqslant \varphi(y) & \text{if } y > 0 \end{array}\right. \tag{2.1}$$

Define a function $\varphi(y)$ as follows:

$$\varphi(y) = \begin{cases} -(2y)^{3/2}/3 & \text{if } 0 \leqslant y \leqslant 1/2 \\ 1/24 - y/2 - y^2/2 & \text{if } y \geqslant 1/2 \end{cases} \tag{2.2}$$

It is not difficult to verify that these relations define $\varphi(y)$ as a smooth function, decreasing monotonically from 0 to $-\infty$ over the non-negative real line $y \in [0, \infty)$. At the point $y = 1/2$ we have $\varphi(y) = -1/3$. $\varphi'(y) = -1$.

The curves Γ and Γ' defined for $y \geqslant 0$ and $y \leqslant 0$ by the formulae $x = \varphi(y)$ and $x = -\varphi(-y)$, respectively, are shown in Fig. 4 (thicker curves). We also show on these curves, which are symmetrical to one another about the origin, the points $A = (-1/3, 1/2)$ and $A' = (1/3, -1/2)$ at which the sections defined by formulae (2.2) meet smoothly.

The curves Γ and Γ' form the boundary of the domain D; according to (2.1), Γ, which lies in the second quadrant of the xy plane, belongs to D, while Γ', which lies in the fourth quadrant, is not contained in D. The union of the domain D with the domain D', symmetric to it with respect to the origin, gives the whole xy plane punctured at the origin O. By (1.9), O is the final point and is therefore of no interest as an initial point: if $x = y = 0$ at time $t = 0$, a control process is needless.

Below we will construct a control and trajectories, that is, functions $u(t), x(t), y(t), z(t)$, for all initial points $(x_0, y_0) \in D$. But if $(x_0, y_0) \in D'$, the required solution will be given by functions $\{-u(t), -x(t), -y(t), -z(t)\}$,

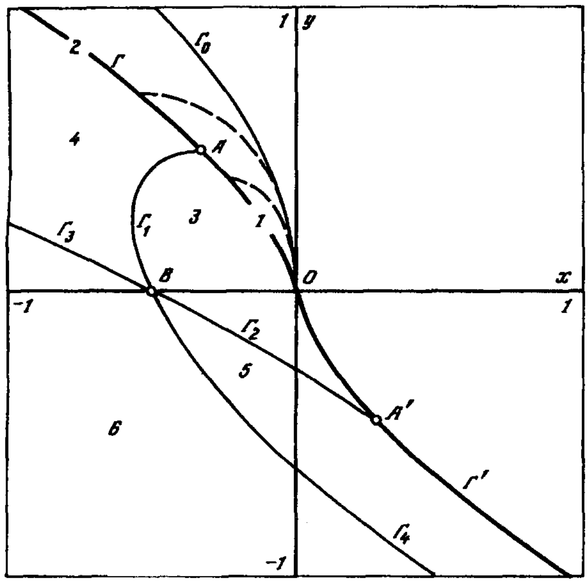

Fig. 4.

where the triple $\{u(t), x(t), y(t), z(t)\}$ is the solution for the initial point $(-x_0, -y_0) \in D$ symmetric to $(x_0, y_0) \in D'$. Then Eqs (1.6) and conditions (1.7) will be satisfied, the trajectories emanating from $(x_0, y_0) \in D'$ will be symmetric to the trajectories from the point $(-x_0, -y_0) \in D$ and will also lead to the origin—moreover, in the same time.

Thus, it will suffice to solve the control problem as formulated for an initial point $(x_0, y_0) \in D$. This will be done with the help of the variations 1–6 shown in Fig. 3.

3. CONSTRUCTION OF THE TRAJECTORIES

We will construct appropriate trajectories for each of variations 1–6 in Fig. 3 and determine the domains D_i $(i = 1, \ldots, 6)$ of initial data x_0, y_0 in the domain D from which the variation in question steers the system to the final state $x(T) = y(T) = 0$. The sets D_i are indicated in Fig. 4 by the corresponding digits $i = 1, \ldots, 6$.

Variation 1. By Fig. 3, $u = -1$ for $t \in [0, T]$. Integrating Eqs (1.6) for initial data (1.8), we obtain

$$u = -1, \quad z = -t, \quad y = y_0 - t^2/2, \quad x = x_0 + y_0 t - t^3/6 \tag{3.1}$$

Set $t = T$ in formulae (3.1) and substitute the results into the final conditions (1.9). Eliminating T, we obtain

$$T = (2y_0)^{1/2} < 1 \tag{3.2}$$

$$x_0 = -(2y_0)^{3/2}/3 \tag{3.3}$$

The inequality $T < 1$ follows from the fact that the bound $z = -1$ is not achieved in variation 1; see Fig. 3. Relations (3.2) and (3.3) imply the inequalities

$$-1/3 < x_0 < 0, \quad 0 < y_0 < 1/2 \tag{3.4}$$

Thus, variation 1, defined by (3.1), is implemented if the initial point (x_0, y_0) lies on an arc of the curve defined by Eq. (3.3) and inequalities (3.4). Consequently, the set D_1 is the arc of the curve Γ (see (2.2)) indicated by the numeral 1 in Fig. 4, enclosed between the points O and $A = (-1/3, 1/2)$. All phase trajectories beginning on this arc will ultimately lead to the origin if variation 1 is applied. The trajectories are defined by (3.1) and the duration of the motion by (3.2). It is easily verified that all these trajectories lie in the domain between the curve Γ and the parabola Γ_0 defined by the formulae

$$\Gamma_0 : x = \varphi_0(y) = -y^2/2, \quad y \geqslant 0 \tag{3.5}$$

This parabola Γ_0 is at the same time the switching curve and a phase trajectory leading to the origin for the time-optimal problem, if conditions (1.7) are replaced by the simple restriction $|z| \leqslant 1$ [1].

Variation 2. We have

$$\begin{aligned} &u = -1, \ z = -t \ \text{ if } \ 0 \leqslant t < 1 \\ &u = 0, \ z = -1 \ \text{ if } \ 1 < t < T \end{aligned} \tag{3.6}$$

Motion along the first part of the trajectory ($t < 1$) is defined by relationships (3.1). We conclude from (3.1) that at $t = 1$

$$y(1) = y_0 - 1/2, \quad x(1) = x_0 + y_0 - 1/6 \tag{3.7}$$

Integrating Eqs (1.6), taking (3.6) and initial data (3.7) in the second part of the motion ($t > 1$) into account, we obtain

$$y(t) = y(1) - (t-1), \quad x(t) = x(1) + y(1)(t-1) - (t-1)^2/2 \tag{3.8}$$

Substitute expressions (3.8) into terminal conditions (1.9) and eliminate T. We obtain

$$T = y(1) + 1 > 1, \quad x(1) = -[y(1)]^2/2 \tag{3.9}$$

Thus, the point $(x(1), y(1))$ lies on the parabola Γ_0 of (3.5), and the second part of the motion (3.6) ($t \in [1, T]$) takes place along this parabola until the origin is reached. Substituting (3.7) into (3.9), we obtain the conditions

$$x_0 = -y_0^2/2 - y_0/2 + 1/24, \quad y_0 \geqslant 1/2 \tag{3.10}$$

Formulae (3.10) define the set D_2 of initial data for which variation 2 ensures that the system will reach the origin. This set D_2 is the part of the curve Γ (see (2.2)) from the point $A = (-1/3, 1/2)$ inclusive to infinity. All trajectories starting in that set are enclosed between Γ and Γ_0, with their second parts (for $t > 1$) lying on the parabola Γ_0. Typical trajectories for variations 1 and 2 are shown in Fig. 4 by dashed curves.

Thus, if the initial point (x_0, y_0) lies on the curve Γ, our problem is solved by controls 1 and 2, with variation 1 applying if (x_0, y_0) is between O and A, and variation 2 if it is to the left of A in Fig. 4.

We now consider variations 3–6 of Fig. 3, letting θ denote the time at which the function $z(t)$, $\theta \in (0, T)$, experiences a jump. It is not difficult to see that the functions $z(t)$ for $t > \theta$ for all variations 3–6 of Fig. 3 are identical with $z(t)$ for $t > 0$ for one of variations 1 or 2: for variations 3 and 5 the relevant variation is 1, and for variations 4 and 6 it is 2. Hence the segments of trajectories for variations 3–6 for $t > \theta$ coincide with trajectories for one of variations 1 or 2. Consequently, the point $(x(\theta), y(\theta))$ for variations 3–6 must belong to the sets of initial data for the appropriate variations 1 or 2, namely

$$\begin{aligned} &(x(\theta), y(\theta)) \in D_1 \ \text{for variations 3, 5} \\ &(x(\theta), y(\theta)) \in D_2 \ \text{for variations 4, 6} \end{aligned} \tag{3.11}$$

To compute the numbers $x(\theta)$ and $y(\theta)$, we note that, apart from sign, the variation of $z(t)$ for $t < \theta$ in cases 3 and 4 is identical with variation 1, and in cases 5 and 6—with variation 2. Therefore, changing signs when needed and setting $t = \theta$, we conclude from (3.11) that for variations 3 and 4

$$y(\theta) = y_0 + \theta^2/2, \quad x(\theta) = x_0 + y_0\theta + \theta^3/6, \quad \theta < 1 \tag{3.12}$$

Using formulae (3.7) and (3.8) and proceeding in analogous fashion, we obtain the results for variations 5 and 6

$$y(\theta)=y_0+\tfrac{1}{2}+(\theta-1)=y_0+\theta-\tfrac{1}{2}$$

$$x(\theta)=x_0+y_0+\tfrac{1}{6}+(y_0+\tfrac{1}{2})(\theta-1)+(\theta-1)^2/2=$$

$$=x_0+y_0\theta+\theta^2/2-\theta/2+\tfrac{1}{6},\quad \theta\geqslant 1 \tag{3.13}$$

Let us determine the domains D_i in the xy plane containing the initial data x_0, y_0 for the appropriate variations, $i = 3, 4, 5, 6$. To do this, we use formulae (3.11)–(3.13) and the previously presented definitions of D_1 and D_2.

Variation 3. Substituting expressions (3.12) for $x(\theta), y(\theta)$ in place of x_0, y_0 in Eq. (3.3) and inequalities (3.4) defining the set D_1, we obtain

$$x_0=-y_0\theta-\theta^3/6-(2y_0+\theta^2)^{3/2}/3 \tag{3.14}$$

$$0<2y_0+\theta^2<1,\quad 0<\theta<1$$

Let us determine the boundaries of the set D_3 defined parametrically by formulae (3.14). To do this, it will suffice to consider four cases, corresponding to equality in each of the four inequalities (3.14).

We first assume that $2y_0 + \theta^2 = 0$.

Substituting the value of θ found from this equality into (3.14), we obtain

$$x_0=(-2y_0)^{3/2}/3,\quad -\tfrac{1}{2}<y_0<0 \tag{3.15}$$

By (2.1) and (2.2), formulae (3.15) define a segment of the curve Γ' from the origin to the point $A' = (1/3, 1/2)$ (see Fig. 4).

Putting $2y_0 + \theta^2 = 1$ and substituting the value of θ thus determined into (3.14), we obtain

$$x_0=-\tfrac{1}{3}-y_0(1-2y_0)^{1/2}-(1-2y_0)^{3/2}/6,\quad 0<y_0<\tfrac{1}{2} \tag{3.16}$$

Formulae (3.16) define an arc of a curve Γ_1 in the xy plane joining the points $A = (-1/3, 1/2)$ and $B = (-1/2, 0)$. This curve is shown in Fig. 4.

Setting $\theta = 0$, we obtain from (3.14)

$$x_0=-(2y_0)^{3/2}/3,\quad 0<y_0<\tfrac{1}{2}$$

By (2.2), this segment of the boundary of D_3, coincides with the set D_1, that is, with the arc OA of the curve Γ.

Finally, setting $\theta = 1$, we obtain from (3.14)

$$x_0=-\tfrac{1}{6}-y_0-(2y_0+1)^{3/2}/3,\quad -\tfrac{1}{2}<y_0<0 \tag{3.17}$$

These formulae define an arc of a curve in the xy plane joining the points B and A'. This curve Γ_2 touches the curve Γ' at the point A' (see Fig. 4).

Thus, the set D_3 is a curvilinear quadrilateral $OABA'$ bounded by arcs of the curves Γ (from O to A), Γ_1, Γ_2 and Γ' (from A' to O).

Variation 4. Substituting expressions (3.12) for $x(\theta), y(\theta)$ in plane of x_0, y_0 in relations (3.10) defining the set D_2, we obtain

$$x_0=\tfrac{1}{24}-y_0^2/2-y_0\theta^2/2-y_0\theta-y_0/2-\theta^4/8-\theta^3/6-\theta^2/4$$

$$2y_0+\theta^2\geqslant 1,\quad 0<\theta<1 \tag{3.18}$$

The boundaries of the set D_4 will be found by replacing the inequality sign in each of the three inequalities of (3.18) in turn by an equality sign.

We first assume that $2y_0 + \theta^2 = 1$, and eliminate θ: $\theta = (1-2y_0)^{1/2}$ from the given equality. Substituting the resulting value of θ into (3.18) and simplifying, we obtain the relation defining the arc Γ_1.

Setting $\theta = 0$ in (3.18), we obtain, as is easily verified, Eqs (3.10), which define the set D_2, that is, the arc of the curve Γ from the point A to infinity.

Setting $\theta = 1$ in (3.18), we have

$$x_0 = -y_0^2/2 - 2y_0 - 1/2, \quad y_0 \geqslant 0 \tag{3.19}$$

The curve Γ_3 defined by these relations begins at the point $B = (-1/2, 0)$ and goes off to infinity (see Fig. 4).

As a result, the set D_4 is bounded by the set D_2, the curve Γ_1—along which it borders on D_3—and the curve Γ_3.

Variation 5. Substituting expressions (3.12) for $x(\theta), y(\theta)$ in place of x_0, y_0 in Eq. (3.3) and inequalities (3.4) defining the set D_1, we obtain

$$\begin{aligned} &x_0 = -y_0\theta - \theta^2/2 + \theta/2 - 1/6 - (2y_0 + 2\theta - 1)^{3/2}/3 \\ &1/2 < y_0 + \theta < 1, \quad \theta \geqslant 1 \end{aligned} \tag{3.20}$$

We now determine the boundaries of the set D_5, reasoning by analogy with the previous cases and replacing each of the three inequalities (3.20) in turn by equalities.

Setting $y_0 + \theta = 1/2$, we ofind that $\theta = 1/2 - y_0$. Substituting this value of θ into equality (3.20), we obtain

$$x_0 = y_0^2/2 - y_0/2 - 1/24, \quad y_0 \leqslant -1/2$$

By (2.2), these formulae define the arc of the curve Γ' from the point $A' = (1/3, -1/2)$ to infinity, this arc is symmetric to D_2 about the origin.

Setting $y_0 + \theta = 1$, we obtain $\theta = 1 - y_0$. Substituting this value into equality (3.20), we obtain

$$x_0 = y_0^2/2 - y_0/2 - 1/2, \quad y_0 \leqslant 0 \tag{3.21}$$

Formulae (3.21) define a curve Γ_4 beginning at $B = (-1/2, 0)$ and going off to infinity (see Fig. 4).

Setting $\theta = 1$ in (3.20), we obtain formulae (3.17) defining the curve Γ_2.

Thus, the set D_5 is bounded by an arc of the curve Γ_2—along which it borders on the set D_3—the curve Γ_4 and the arc of the curve Γ' from A' to infinity.

Variation 6. Substituting expressions (3.13) for $x(\theta), y(\theta)$ in place of x_0, y_0 in formulae (3.10) defining the set D_2, we obtain

$$x_0 = -y_0^2/2 - 2y_0\theta - \theta^2 + \theta/2, \quad y_0 + \theta \geqslant 1, \quad \theta \geqslant 1 \tag{3.22}$$

Replacing the first of inequalities (3.22) by an equality, we obtain $\theta = 1 - y_0$. Substituting this expression into (3.22), we obtain relations (3.21) defining the curve Γ_4.

Setting $\theta = 1$ in (3.22), we obtain relations (3.19) defining the curve Γ_3.

Thus, the set D_6 borders on the sets D_4 and D_5 along the curves Γ_3 and Γ_4, respectively, and lies below and to the left of these curves, which have a common point B $(-1/2, 0)$.

Note that the curves Γ_2 and Γ_3 have a common tangent at the point B, and the same is true of Γ_3 and Γ_4.

4. CONCLUSION

The solution of the control problem as formulated may be described as follows. Given an initial state (1.8) in the domain D of the xy plane, we determine to which of the domains D_i ($i = 1, 2, \ldots, 6$) it belongs. The boundaries between the domains are given by the curves $\Gamma, \Gamma', \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ defined by formulae (2.2), (3.16), (3.17), (3.19) and (3.21). The boundary between D_1 and D_2 is the point $A = (-1/3, 1/2)$.

1. If $(x_0, y_0) \in D_1$, then we define $u \equiv -1$ for $t > 0$. The system reaches the given state $x = y = 0$ in a time $T < 1$.

2. If $(x_0, y_0) \in D_2$, we put $u = -1$ for $t \in (0, 1)$ and $u = 0$ for $t \geqslant 1$. The system reaches the final state in a time $T \geqslant 1$.

3. If $(x_0, y_0) \in D_3$, then $u = 1$ for $t \in (0, \theta)$, where the time $\theta < 1$ is defined by the condition $(x(\theta), y(\theta)) \in D_1$. At time $t = \theta$ we equate z to zero by a jump, which is admitted by restrictions (1.7). At $t > \theta$ we define $u = -1$ up to the end of the process. The trajectory for $t > \theta$ is the same as for variation 1.

4. If $(x_0, y_0) \in D_4$, then $u = 1$ for $t \in (0, \theta)$, where the time $\theta < 1$ is defined by the condition $(x(\theta), y(\theta)) \in D_2$. At time $t = \theta$ we equate z to zero by a jump. We then define $u = -1$ for $t \in (\theta, \theta + 1)$ and $u = 0$ for $t \in (\theta + 1, T)$.

5. If $(x_0, y_0) \in D_5$, then $u = 1$ for $t \in (0, 1)$ and $u = 0$ for $t \in (1, \theta)$, where the time $\theta > 1$ is defined by the condition $(x(\theta), y(\theta)) \in D_1$. At time θ we equate z to zero by a jump. We then define $u = -1$ for $t \in (\theta, T)$ up to the end of the process.

6. If $(x_0, y_0) \in D_6$, then $u = 1$ for $t \in (0, 1)$ and $u = 0$ for $t \in (1, \theta)$, where the time $\theta > 1$ is defined by the condition $(x(\theta), y(\theta)) \in D_2$. At time θ we equate z to zero by a jump. We then define $u = -1$ for $t \in (\theta, \theta + 1)$ and $u = 0$ for $t \in (\theta + 1, T)$.

Note that $T < 1$ in case 1, $T > 1$ in cases 2, 4, 5, and $T > 2$ in case 6.

All trajectories beginning in the domain D lie in the domain bounded by the curves Γ_0 and Γ' (to the left of and below those curves; see Fig. 4). They reach the origin O either touching the curve Γ_0 (for laws 1, 3, and 5) or coinciding with Γ_0 over its last part (for variations 2, 4 and 6); see the curves in Fig. 4.

If the initial point (x_0, y_0) is in the domain D' symmetric to D about the origin, the control is taken equal in magnitude and opposite in sign to the control corresponding to the point $(-x_0, -y_0) \in D$.

The solution we have constructed was obtained for initial data (1.8), which presume that $z(0) = 0$. The general case of initial data

$$x(0) = x_0, \quad y(0) = y_0, \quad z(0) = z_0 \tag{4.1}$$

is reduced to that considered above if at time $t = 0$ we change z by a jump, equating it to zero, which is admitted by restrictions (1.7). Thus, the solution constructed above is suitable for the case of general initial data (4.1).

In that case, however, it loses the property of time-optimality.

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